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Coercivity properties for monotone functionals

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Abstract. For a monotone functional defined only on a closed convex cone in a quasi-ordered Banach space it is shown that a version of Palais–Smale condition adapted to the quasi-order structure implies a conical coercivity property. The conical asymptotic behavior of monotone functionals is also studied.

Keywords: Quasi-order, variational principle, convex cone, monotone functional, directional derivative, coercivity, Palais–Smale condition.

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Introduction

Let $(X, \|\cdot\|)$ be a real Banach space and let X^* denote its dual. Given a functional $f\colon X \to \mathbb{R}$, we recall that f is coercive if

$$f(u) \to +\infty \quad \text{as} \quad \|u\| \to +\infty. \tag{1}$$

Another basic concept is the one of Palais–Smale condition. A Gâteaux differentiable functional $f\colon X \to \mathbb{R}$ is said to satisfy the Palais–Smale condition if every sequence (v_n) in X such that $f(v_n)$ is bounded and $f'(v_n) \to 0$ in X^* as $n \to +\infty$ contains a subsequence which is strongly convergent in X.

The relationship between the coercivity property and the Palais–Smale condition was extensively studied. A typical result in this direction is the one due to Caklovic, Li and Willem [2] stating that under the assumption

$$f \text{ is Gâteaux differentiable and lower semicontinuous (in short l.s.c.),} \tag{2}$$

the Palais–Smale condition and the boundedness from below imply the coercivity. When $f \in C^1(X)$ this result was also established in Brézis and Nirenberg [1].

An extension of the result in [2] for functionals satisfying (2) has been obtained by Goeleven [4]. Specifically, the functional considered in [4] is required to be of the form $f = g + h$, with

$$g \text{ is Gâteaux differentiable, l.s.c. and } h \text{ is proper, convex, l.s.c.,} \tag{3}$$

where an appropriate concept of Palais–Smale condition was introduced. A version of this result was given by D. Motreanu and V. V. Motreanu [5] for the case where f is supposed to admit the decomposition $f = g + h$ with

$$g \text{ is locally Lipschitz and } h \text{ is proper, convex, l.s.c.} \tag{4}$$

and the Palais–Smale condition given in [7], Chapter 3.

Notice that the l.s.c. property is crucial for all these developments (see (2), (3), (4)). A natural question is if this property can be weakened but still preserving the nature of the results above. As shown in D. Motreanu, V. V. Motreanu and M. Turinici [6], this question is answered in the affirmative in a quasi-ordered context. Precisely, if X is endowed with a quasi-order $\leq_K$ generated by a closed convex cone K, replacing the l.s.c. property by

$$f \text{ is } \leq_K\text{-l.s.c.} \tag{5}$$

one shows that the Palais–Smale condition as formulated in [6] implies the coercivity.

A relevant case in (5) is the situation where f is $\leq_K$-decreasing. Unfortunately, for such a functional the Palais–Smale condition used in [6] does not hold and so for a $\leq_K$-decreasing functional f the result in [6] is not applicable.

Since for a $\leq_K$-decreasing (or, equivalently, $\leq_{(-K)}$-increasing) functional the global coercivity is not true, we can expect to have the coercivity on a prescribed cone. The object of this paper is to study this conical coercivity. Our result in this direction is stated in Theorem 4. This will be deduced from our main result given in Theorem 3 which presents the asymptotic behavior of a functional defined only on a cone. The basic tool for our main result is the order version of Ekeland's variational principle (see [3]) established in Turinici [8].

The rest of the paper is organized as follows. Section 1 is devoted to recalling the monotone variational principle. Section 2 contains our main result. In Section 3 a suitable order version of Palais–Smale condition and our conical coercivity result are given.

1 Monotone variational principle

Let (M, d) be a complete metric space endowed with a quasi-order $\leq$ (i.e., a reflexive and transitive relation on M) and let $f\colon M \to \mathbb{R}$ a function. We say that the quasi-order $\leq$ is *closed from the left* if $\{y \in M \mid \ y \leq x\}$ is closed, for each $x \in M$. The functional f is called $\leq$-*increasing* (in short, increasing) if $x, y \in M,\ x \leq y \Rightarrow f(x) \leq f(y)$. The quasi-order $\leq$ is said to be *closed from the right* if the dual quasi-order $\geq$ is closed from the left. The functional f is said $\leq$-*decreasing* (in short, decreasing) if $-f$ is $\leq$-increasing.

The following result can be viewed as the monotone counterpart of Ekeland's variational principle [3], where the l.s.c. hypothesis is replaced by a monotone assumption.

Theorem 1 (Turinici [8]). *Assume that the quasi-order $\leq$ on a complete metric space (M, d) is closed from the right and let $f\colon M \to \mathbb{R}$ be a function which is bounded from below and decreasing. Then, for each $\eta > 0$ and $u \in M$ there exists $v = v(\eta, u) \in M$ with*

$$u \leq v,\ \eta d(u, v) \leq f(u) - f(v), \tag{6}$$

$$w \in M,\ v \leq w,\ w \neq v \Rightarrow \eta d(v, w) > f(v) - f(w). \tag{7}$$

Remark 1. Ekeland's variational principle [3] for real-valued functionals can be obtained from Theorem 1 as follows. If (M, d) is a complete metric space and $f\colon M \to \mathbb{R}$ is bounded from below and l.s.c. on M, for a fixed $\eta > 0$ we define

$$x \leq y \iff \eta d(x, y) \leq f(x) - f(y).$$

Clearly, $\leq$ is an (antisymmetric) quasi-order on M which is closed from the right due to the l.s.c. assumption upon f. By the definition of quasi-order $\leq$ it is seen that f is decreasing. Hence Theorem 1 applies ensuring that for each $u \in M$ we find $v = v(\eta, u) \in M$ satisfying (6) (which is just as in Ekeland's variational principle) and (7). In order to get the conclusion of Ekeland's variational principle it remains to show that

$$w \in M,\ w \neq v \Rightarrow \eta d(v, w) > f(v) - f(w).$$

Arguing by contradiction assume that there exists $w \in M \setminus \{v\}$ such that

$$\eta d(v, w) \leq f(v) - f(w) \tag{8}$$

which is equivalent, according to the definition of $\leq$, to $v \leq w$. Then (7) leads to a contradiction with (8).

In the next Section the reformulation of Theorem 1 for increasing functions is needed.

Theorem 2. *Suppose that the quasi-order $\leq$ on a complete metric space (M, d) is closed from the left and let $f\colon M \to \mathbb{R}$ be a function which is bounded* *from below and increasing. Then, for each* $\eta > 0$ *and* $u \in M$ *there exists* $v = v(\eta, u) \in M$ *with*

$$v \leq u,\ \eta d(u,v) \leq f(u) - f(v), \tag{9}$$

$$w \in M,\ w \leq v,\ w \neq v \Rightarrow \eta d(v,w) > f(v) - f(w). \tag{10}$$

PROOF. It is sufficient to apply Theorem 1 for the quasi-order $\geq$ (the dual of the quasi-order $\leq$) on M and the function f. $\boxed{QED}$

2 Main result

Let $(X, \|\cdot\|)$ be a real Banach space and let K be a closed convex cone, i.e. K is a closed subset of X with the properties

$$K + K \subseteq K; \quad \lambda K \subseteq K,\ \forall \lambda \geq 0.$$

The relation $\leq_K$ on X defined by

$$x \leq_K y \iff y - x \in K \tag{11}$$

is a quasi-order on X. Given $u \in K$ we introduce the subset $H(u)$ of $-K$ as follows

$$H(u) = \{h \in -K \mid\ u + \lambda h \in K \text{ for some } \lambda > 0\}. \tag{12}$$

We see that $H(u)$ is a convex subcone of $-K$ with $-u \in H(u)$.

For a function $f\colon K \to \mathbb{R}$ and the elements $u \in K$, $h \in H(u)$ we define the h-directional derivative of f at u by

$$f'(u;h) = \lim_{t \to 0^+} \frac{1}{t}(f(u+th) - f(u)),$$

whenever this limit exists in $\overline{\mathbb{R}}$. We point out that the limit above makes sense since $u + th \in K$ for $t > 0$ sufficiently small. Indeed, if $u \in K$ and $h \in H(u)$ then $u + \lambda h \in K$ for some $\lambda > 0$, therefore

$$\frac{\lambda}{t} u + \lambda h = \left(\frac{\lambda}{t} - 1\right) u + (u + \lambda h) \in K + K \subseteq K, \quad \text{for all } t \in]0, \lambda[,$$

or, equivalently, $u + th \in K$ for $0 < t < \lambda$. If there exists $f'(u;h)$ for all $h \in H(u)$, we say that f has a $H(u)$-directional derivative at u.

For each $\rho > 0$ denote

$$K_\rho = \{x \in K \mid \|x\| \geq \rho\}, \quad K_\rho^0 = \{x \in K \mid\ \|x\| > \rho\}. \tag{13}$$

In the following we need a hypothesis which is related to the definition of $\leq_K$ in (11): there exists $\rho > 0$ such that

$$f \text{ is } \leq_K\text{-increasing on } K_\rho, \tag{14}$$

$$f \text{ has a } H(u)\text{-directional derivative at each point } u \text{ of } K_\rho^0, \tag{15}$$

$$f \text{ is bounded from below on } K_\rho. \tag{16}$$

Some properties for the $H(u)$-directional derivative are derived in the lemma below.

Lemma 1.

1) *If* (15) *holds for some* $\rho > 0$*, then*

$$\nu(f'(u); H(u)) := \sup\{(-f)'(u; h);\ h \in H(u),\ \|h\| = 1\} \tag{17}$$

is well defined (possibly equal to $+\infty$*) for each* $u \in K_\rho^0$*.*

2) *If* (14) *and* (15) *hold for some* $\rho > 0$*, one has that*

$$(-f)'(u; h) \geq 0, \quad \forall u \in K_\rho^0,\ \forall h \in H(u),$$

so,

$$0 \leq \nu(f'(u); H(u)) \leq +\infty, \quad \forall u \in K_\rho^0.$$

Proof.

1) Fix $u \in K_\rho^0$. The set of constraints $\{h \in H(u);\ \|h\| = 1\}$ is nonempty because $-u \in H(u)$ and $u \neq 0$. Assumption (15) ensures that there exists the $H(u)$-directional derivative of $-f$ at u in $\overline{\mathbb{R}}$.
2) For fixed elements $u \in K_\rho^0$ and $h \in H(u)$ we have that $u + th \in K_\rho$ if $t > 0$ is small enough and, naturally, $u + th \leq_K u$. Applying condition (14) one obtains $(-f)'(u; h) \geq 0$. Using (15), by part 1) we get also the last assertion.

$\boxed{QED}$

The main result of this Section is the following.

Theorem 3. *Let* K *be a closed convex cone in the Banach space* X *and let* $f\colon K \to \mathbb{R}$ *be a functional satisfying* (14), (15), (16) *(for some* $\rho > 0$*) together with*

$$\alpha := \liminf_{\|u\| \to +\infty} f(u) < +\infty. \tag{18}$$

Then for every sequence $(\varepsilon_n) \subset \mathbb{R}^+$ *with* $\varepsilon_n \to 0^+$ *there exists a sequence* $(v_n) \subset K_\rho^0$ *(see* (13)*) satisfying*

$$\|v_n\| \to +\infty, \;\; f(v_n) \to \alpha \quad as \quad n \to +\infty \tag{19}$$

and

$$0 \leq \nu(f'(v_n); H(v_n)) \leq \varepsilon_n, \quad for\ every\ n. \tag{20}$$

Remark 2. We stress that in (18) the limit is for $u \in K$ (here f is defined only on K). The number $\nu(f'(v_n); H(v_n))$ in Theorem 3 exists by part 1) of Lemma 1 and is known to be positive by part 2) of Lemma 1.

Proof of Theorem 3. Let us note that (18) can be expressed as

$$\alpha = \sup_{\sigma>0} \inf_{u \in K_\sigma} f(u) < +\infty. \tag{21}$$

Denoting

$$m(\sigma) = \inf_{u \in K_\sigma} f(u), \quad \forall \sigma > 0, \tag{22}$$

by (21) and (22) one has

$$\lim_{\sigma \to +\infty} m(\sigma) = \alpha. \tag{23}$$

Fix a sequence $(\varepsilon_n) \subset \mathbb{R}^+$ with $\varepsilon_n \to 0^+$. For each integer $n \geq 1$, equality (23) enables us to determine some $r_n > \max\{n, \rho, 2\varepsilon_n\}$ such that

$$m(\sigma) \geq \alpha - \varepsilon_n^2, \quad \forall \sigma \geq r_n. \tag{24}$$

On the other hand, (21) and (22) imply

$$m(2r_n) \leq \alpha < \alpha + \varepsilon_n^2,$$

wherefrom there exists some point $u_n \in K$ with

$$\|u_n\| \geq 2r_n, \quad f(u_n) < \alpha + \varepsilon_n^2. \tag{25}$$

Note that in view of $r_n > n$ one has

$$r_n \to +\infty \quad \text{as} \quad n \to +\infty. \tag{26}$$

Since

$$\{y \in K_{r_n} \mid \; y \leq_K x\} = K_{r_n} \cap (x + (-K)),$$

the quasi-order $\leq_K$ on K_{r_n} is closed from the left. By (14) and (16) it is clear that Theorem 2 applies with $M = K_{r_n}$, $f = f\big|_{K_{r_n}}$, $\eta = \varepsilon_n$, $u = u_n \in K_{r_n}$ (cf. (25)) and for the quasi-order $\leq_K$. Consequently, there exists a point $v_n \in K_{r_n}$ with

$$v_n \leq_K u_n, \quad \varepsilon_n \|u_n - v_n\| \leq f(u_n) - f(v_n), \tag{27}$$

$$w \in K_{r_n},\ w \leq_K v_n,\ w \neq v_n \Rightarrow \varepsilon_n \|v_n - w\| > f(v_n) - f(w) \tag{28}$$

(see the corresponding relations (9), (10) in Theorem 2). Using $v_n \in K_{r_n}$ and (26), we see that $\|v_n\| \to +\infty$ as $n \to +\infty$. Further, (24), (22), (27) and (25) imply

$$\alpha - \varepsilon_n^2 \leq m(r_n) \leq f(v_n) \leq f(u_n) < \alpha + \varepsilon_n^2. \tag{29}$$

Since $\varepsilon_n \to 0^+$, it follows that $f(v_n) \to \alpha$ as $n \to +\infty$, so (19) is fulfilled.

By (27) and (29) one has

$$\varepsilon_n \|u_n - v_n\| < 2\varepsilon_n^2.$$

Then, taking into account the first part in (25) and $r_n > 2\varepsilon_n$, we infer that

$$\|v_n\| \geq \|u_n\| - \|u_n - v_n\| > 2r_n - 2\varepsilon_n > r_n. \tag{30}$$

It turns out that $v_n \in K_\rho^0$ since $r_n > \rho$.

Let $h \neq 0$ be an arbitrary element of $H(v_n)$ (see (12)). There exists $\tau_n = \tau_n(h) > 0$ such that $v_n + \tau_n h \in K$. The convexity of K entails $v_n + th \in K$, $\forall t \in [0, \tau_n]$. According to (30), for a possibly smaller $\tau_n > 0$ we may suppose that

$$\|v_n + th\| > r_n, \quad \forall t \in [0, \tau_n].$$

Moreover, for all $t \in [0, \tau_n]$ we have $v_n + th \leq_K v_n$, because $h \in H(v_n)$. Therefore, for all $t \in]0, \tau_n]$ we may set $w = v_n + th$ in (28). This yields

$$0 \geq \frac{1}{t}(f(v_n + th) - f(v_n)) > -\varepsilon_n \|h\|, \quad \forall t \in]0, \tau_n].$$

Letting $t \to 0^+$ implies $0 \geq f'(v_n; h) \geq -\varepsilon_n \|h\|$. In view of (17), property (20) follows. The proof is complete. $\boxed{QED}$

Remark 3. Theorem 3 is applicable for any ρ larger than the one used in its statement. This is due to the hereditary character of conditions (14), (15), (16).

3 Conical coercive functionals

In this Section we study the coercivity only on a cone, which is clearly weaker than the coercivity on the whole space in (1).

Throughout the rest of the paper K stands for a closed convex cone in the Banach space X.

We introduce a variant of Palais–Smale condition for a nonsmooth functional which is defined only on the cone K.

Definition 1. A function $f\colon K \to \mathbb{R}$ satisfies *the Palais–Smale condition with respect to* K *and the number* $\rho > 0$ if there exists a sequence $(\varepsilon_n) \subset \mathbb{R}^+$ with $\varepsilon_n \to 0^+$ such that each sequence $(v_n) \subset K$ verifying

(i) $\|v_n\| > \rho$,

(ii) $f(v_n)$ bounded,

(iii) $\nu(f'(v_n); H(v_n))$ (see (17)) exists,

(iv) $0 \le \nu(f'(v_n); H(v_n)) \le \varepsilon_n$,

for all n, contains a (strongly) convergent subsequence.

Remark 4. The Palais–Smale condition with respect to a closed convex cone K and a number $\rho > 0$ formulated in Definition 1 is much weaker than the usual Palais–Smale condition. Clearly, if $f\colon X \to \mathbb{R}$ is a differentiable functional satisfying the ordinary Palais–Smale condition then Definition 1 is verified for $K = X$ and every $\rho > 0$.

The coercivity along the cone is stated in the following.

Theorem 4. *Let K be a closed convex cone in the Banach space X and let the functional $f\colon K \to \mathbb{R}$ satisfy* (14)*,* (15)*,* (16) *(for some $\rho > 0$) as well as the Palais–Smale condition with respect to K and ρ in Definition 1. Then f is coercive (on K), i.e.* (1) *holds for $u \in K$.*

Proof. Arguing by contradiction, assume that (18) holds. Let $(\varepsilon_n) \subset \mathbb{R}^+$ with $\varepsilon_n \to 0^+$ be the sequence given by Definition 1. The imposed assumptions allow to invoke Theorem 3. Applying Theorem 3 for (ε_n), a sequence $(v_n) \subset K_\rho^0$ is found such that (19) and (20) hold. Hence conditions (i)-(iv) in Definition 1 are verified. Then the Palais–Smale condition with respect to K and ρ guarantees that (v_n) contains a strongly convergent subsequence. This contradicts the first part of (19), which completes the proof. $\boxed{QED}$

Remark 5. The fact that in the statement of Theorem 4 we assumed that (14), (15), (16) and Definition 1 are satisfied with the same $\rho > 0$ is not restrictive in view of the hereditary property pointed out in Remark 3.

Remark 6. Theorem 4 represents a conical version of the coercivity results studied in other papers (Brézis and Nirenberg [1], Caklovic, Li and Willem [2], Goeleven [4], D. Motreanu and V. V. Motreanu [5]). The reason for the $\le_K$-monotonicity assumption is to compensate the difficulty of working only on a cone K and with a much weaker form of Palais–Smale condition. Contrasting to [6], here the $\le_K$-lower semicontinuity is not involved.

Remark 7. The idea to consider in our setting functionals $f\colon K \to \mathbb{R}$ defined only on a closed convex cone K is inspired from a referee's suggestion for a previous version of the paper to deal with $f\colon K \cup K^- \cup \{0\} \to \mathbb{R}$.

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